Kruskal's Tree Theorem for Term Graphs

Georg Moser **Maria A Schett**

TERMGRAPH 2016, Eindhoven

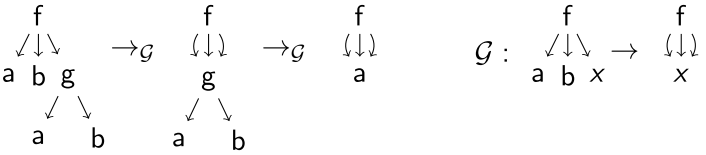
April 8, 2016

(our) background: **term rewriting** $\mathcal{R}$: $\mathsf{f}(\mathsf{a}, \mathsf{b}, x) \to \mathsf{f}(x, x, x)$

$\mathsf{g}(x, y) \to x \quad \mathsf{g}(x, y) \to y$

$$\mathsf{f}(\mathsf{a}, \mathsf{b}, \mathsf{g}(\mathsf{a}, \mathsf{b})) \to_{\mathcal{R}} \mathsf{f}(\mathsf{g}(\mathsf{a}, \mathsf{b}), \mathsf{g}(\mathsf{a}, \mathsf{b}), \mathsf{g}(\mathsf{a}, \mathsf{b})) \to_{\mathcal{R}}^{2} \mathsf{f}(\mathsf{a}, \mathsf{b}, \mathsf{g}(\mathsf{a}, \mathsf{b}))$$

term graph rewriting to simulate term rewriting



$\Longrightarrow$ directed, acyclic, first-order term graphs

termination technique
for term graph rewriting

related work interpretion methods:

- Proving Termination of Graph Transformation Systems Using Weighted Type Graphs over Semirings, Bruggink et al, 2015
- Non-simplifying Graph Rewriting Termination, Bonfante et al, 2013

inspiration

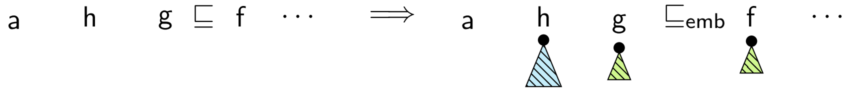
- Simplification Orders for Term Graph Rewriting, Plump, 1997
  - different view on term graphs and embedding
  - different proof of Kruskal's Tree Theorem

- Well-Founded Recursive Relations, Jean Goubault-Larrecq, 2001

for any infinite sequence
$\exists\ i < j$ s.t. $a_i \sqsubseteq_{\mathsf{emb}} a_j$

Kruskal's Tree Theorem

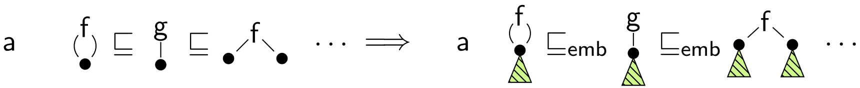
wqo $\sqsubseteq$ on "function symbols" $\Longrightarrow$ wqo $\sqsubseteq_{\mathsf{emb}}$ on "terms"



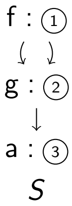
Kruskal's Tree Theorem for Term Graphs

termination technique
based on orders

wqo on "tops" $\Longrightarrow$ wqo on "term graphs"



Term Graphs (formally)



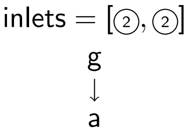
term dag S : $(N, \text{label}, \text{succ})$, $n \in S$

- nodes $N \subseteq \mathbb{N}$
- label : $N \to \mathcal{F}$unction symbols $\cup$ $\mathcal{V}$ariables
- succ : $N \to N^*$ $\quad n \rightharpoonup n'$

if $\text{label}(n) \in \mathcal{V}$ then $\text{succ}(n) = [\,]$

else $\text{succ}(n) = [n_1 \ldots n_{\text{arity}(\text{label}(n))}]$

term graph root : $\exists n. n \rightharpoonup^* n'$ for all $n' \in S$



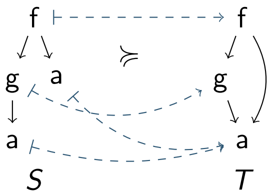
sub graph $S{\restriction}[n_1, \ldots, n_k]$: $\{n \mid n_i \rightharpoonup^* n, 1 \leqslant i \leqslant k\}$

argument graph $S{\restriction}\text{inlets}$

- inlets : $\text{succ}(\text{root}(S))$

misc size: $|S|$ ground: label : $N \to \mathcal{F}$

Sharing



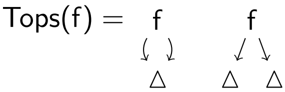
$n \in S$ is *morphic* if

- $\mathsf{label}_S(n) = \mathsf{label}_T(m(n))$
- if $n \xrightarrow{i}_S n_i$ then $m(n) \xrightarrow{i}_T m(n_i)$ for all appropriate i.

morphism $m : S \to T$ is morphic in all $n \in S$

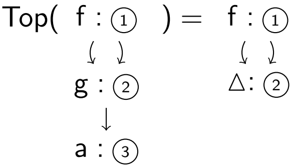
sharing $S \succcurlyeq T$, if exists $m : S \to T$ and $m(\mathsf{root}(S)) = \mathsf{root}(T)$

Tops & Top & Precedence



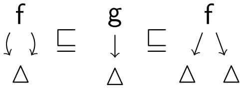
$\mathrm{Tops}(f) = \{ T \mid S \succcurlyeq T \}$

- S = tree representation of $\mathrm{f}(\triangle, \ldots, \triangle)$



$\mathrm{Top}(n) = (N', \mathrm{label}', \mathrm{succ}')$

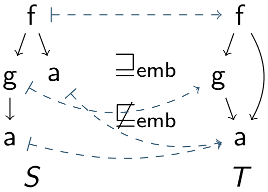
- $N' = n \cup n_i$, where $n_i \in \mathrm{succ}(n)$
- $\mathrm{label}'(n_i) = \triangle$
- $\mathrm{succ}'(n_i) = [\,]$



precedence transitive $\sqsubseteq$

- $S \preccurlyeq T$ and $T \preccurlyeq S$ implies $S \sqsubseteq T$ and $T \sqsubseteq S$
- $T \sqsubseteq S$ implies $|T| \leqslant |S|$.

Embedding



- after $\sqsubseteq_{\text{emb}}$ Plump (1997)
- **idea** take sharing below direct successors into account

new definition of $\sqsubseteq_{\text{emb}}$

$S \sqsupseteq_{\text{emb}} T$ if there exists a partial, surjective function $m \colon S \to T$, s.t. for all nodes s in the domain of S, we have

- $\text{Top}_S(s) \sqsupseteq \text{Top}_T(m(s))$
- $m(s) \rightharpoonup_T m(s')$ implies $s \rightharpoonup_S^+ s'$.

transitivity $\sqsupseteq_{\text{emb}}$ is transitive

$S \sqsupseteq_{\text{emb}} T \sqsupseteq_{\text{emb}} U$ by $m_{S,U}(n) = m_{T,U}(m_{S,T}(n))$

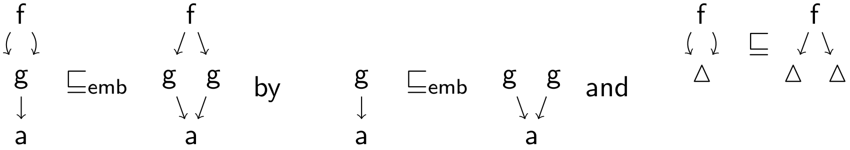
Theorem (Kruskal's, for Term Graphs)

If $\sqsubseteq$ is a wqo on $\mathrm{Tops}(\mathcal{F})$, *then* $\sqsubseteq_{\mathrm{emb}}$ *is a wqo on ground term graphs.*

minimal bad sequence argument

not $\exists\ i < j$ s.t. $a_i \sqsubseteq_{\mathrm{emb}} a_j$

- assume minimal "bad" infinite sequence
- construct even smaller infinite sequence of arguments ("good"!)
- "re-attach tops"

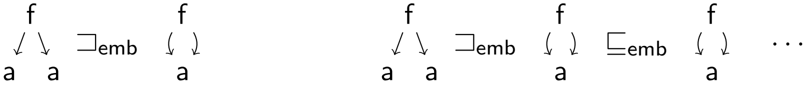


Challenges

liberal definition of $\sqsubseteq_{\text{emb}}$

- $m(s) \rightharpoonup_T m(s')$ implies $s \rightharpoonup_S^+ s'$

simplification order $\succ \supseteq \sqsupset_{\text{emb}}$

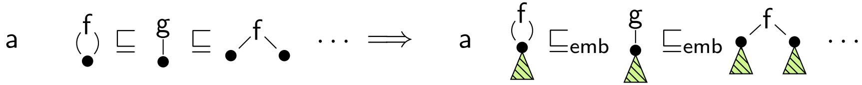


"all steps oriented"

Conclusion

Kruskal's Tree Theorem for Term Graphs

wqo on "tops" $\Longrightarrow$ wqo on "term graphs"



Future Work

- investigate how to enforce order on arguments
- design termination technique based on $\sqsubseteq_{\mathsf{emb}}$
- automation

Thank you for your attention!

Well-Quasi Orders

- A sequence over A is called *good*, if there are $i < j$, such that $a_i \preceq a_j$. Otherwise it is called *bad*.
- A reflexive and transitive order $\succeq$ is a *well-quasi order* (wqo), if every infinite sequence is good.
- A sequence is a *chain*, if $a_i \preceq a_{i+1}$ holds for all $i \geqslant 1$.

Lemma

If $\succeq$ is a wqo then every infinite sequence contains a chain.

Theorem

If $\sqsubseteq$ is a wqo on $\text{Tops}(\mathcal{F})$*, then* $\sqsubseteq_{\text{emb}}$ *is a wqo on ground term graphs.*

proof.

- We construct a minimal bad sequence of term graphs $\mathbf{T}$.
  - Assume we picked $T_1, \ldots, T_{n-1}$.
  - We pick T_n which is minimal wrt. $|N_{T_n}|$, s.t. there are bad sequences that start with $T_1, \ldots, T_n$.
- $G_i = (N_{T_i} \setminus \{\text{root}(T_i)\}, \text{label}_{T_i}|_{N_{G_i}}, \text{succ}_{T_i}|_{N_{G_i}}, \text{succ}_{T_i}(\text{root}(T_i)))$
- $G = \bigcup_{i \geqslant 1} G_i$
- We want to proof $\sqsubseteq_{\text{emb}}$ is a wqo on G.
- Assume G admits a bad sequence $\mathbf{H}$ with $H_1 = G_k$.
- $G' = \bigcup_{i \geqslant 1}^{k} G_i$
- G' is finite, there exists an index $l > 1$, s.t. for all $H_i, i \geqslant l$, $H_i \in G \setminus G'$

proof cont.

- $T_1, \ldots, T_{k-1}, G_k, \mathbf{H}_{\geqslant l}$ is good by minimality of $\mathbf{T}$.
- So we try to find $H_i \sqsubseteq_{\text{emb}} H_j$.

$\underbrace{T_1, \ldots, T_{k-1}}_{i,j}, G_k, \mathbf{H}_{\geqslant l}$ but $H_i = T_i \sqsubseteq_{\text{emb}} T_j = H_j$ ↯

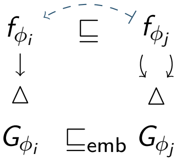
$\underbrace{T_1, \ldots, T_{k-1}}_{i}, \underbrace{G_k}_{j}, \mathbf{H}_{\geqslant l}$ but $H_i = T_i \sqsubseteq_{\text{emb}} G_k = H_j \wedge G_k \sqsubseteq_{\text{emb}} T_k$ ↯

$\underbrace{T_1, \ldots, T_{k-1}}_{i}, G_k, \underbrace{\mathbf{H}_{\geqslant l}}_{j}$ $H_j \notin G'$ but $H_j = G_m \sqsubseteq_{\text{emb}} T_m, m > k$ and $H_i = T_i \sqsubseteq_{\text{emb}} G_m = H_j$ hence $T_i \sqsubseteq_{\text{emb}} T_m$ ↯

- Hence, $H_i \sqsubseteq_{\text{emb}} H_j$ in $G_k, \mathbf{H}_{\geqslant l}$
- ↯ badness of $\mathbf{H}$
- Hence, $\sqsubseteq_{\text{emb}}$ is wqo on G.

proof cont 2.

- $\mathbf{f}$ sequence of Top of $\mathbf{T}$
- $\mathbf{f}$ contains a chain $\mathbf{f}_\phi$, $f_{\phi_i} \sqsubseteq f_{\phi_j}$
- $\sqsubseteq_{\mathsf{emb}}$ is wqo on G_ϕ, hence we have $G_{\phi_i} \sqsubseteq_{\mathsf{emb}} G_{\phi_j}$
- implies $T_{\phi_i} \sqsubseteq_{\mathsf{emb}} T_{\phi_j}$



- ↯ badness of $\mathbf{T}$ □